DISCRETE WAVE-FRONT SETS OF FOURIER LEBESGUE AND MODULATION SPACE TYPES

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Abstract. We introduce discrete wave-front sets with respect to Fourier Lebesgue and modulation spaces. We prove that these wave-front sets agree with corresponding wave-front sets of "continuous type".

0. Introduction

In [12], (continuous) wave-front sets of Fourier Lebesgue and modulation space types were introduced, and the usual mapping properties for pseudo-differential operators were established. Here it was also proved that wave-front sets of Fourier Lebesgue and modulation space types agree with each others, and that the usual wave-front sets with respect to smoothness (cf. [11, Sections 8.1–8.3]) is a special case of wave-front sets of Fourier Lebesgue types. Notice that the analysis of [12] includes pseudo-differential operators with non-smooth symbols. Micro-local analysis of convolution, multiplication and semi-linear equations in Fourier Lebesgue spaces (and therefore modulation spaces as well) can be found in [13].

In this paper we introduce discrete versions of wave-front sets of Fourier Lebesgue and modulation space types, and prove that they coincide with corresponding continuous versions. Furthermore, the established results are formulated in such way that they should be possible to implement in numerical computations. An expected benefit of such approach is that the formulas might serve as an appropriate tool when making numerical analysis of micro-local investigations. For example, we use Gabor frames for the description of discrete wave-front sets. We refer to [5,6] for numerical treatment of Gabor frame theory.

Assume that $p,q \in [1,\infty]$, ω is an appropriate weight function on the phase space $\mathbf{R}^{2d}$ and that f is a distribution defined on the open subset X of $\mathbf{R}^d$. Roughly speaking, the wave-front set $WF_{\mathscr{F}L^q_{(\omega)}}(f)$ with respect to the Fourier Lebesgue space $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ of f, give information about all points $x \in X$ and directions $\xi \in \mathbf{R}^d \setminus 0$ where f locally fails to

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be in $\mathscr{F}L^q_{(\omega)}$. That is $(x,\xi)\in WF_{\mathscr{F}L^q_{(\omega)}}(f)$, if and only if f is locally not in $\mathscr{F}L^q_{(\omega)}$ at x and in the direction ξ. In the same way, the wave-front set of f with respect to the modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, $WF_{M^{p,q}_{(\omega)}}(f)$ consists of all pairs (x,ξ), where f locally at x fails to belong to $M^{p,q}_{(\omega)}$ in the direction ξ.

As a consequence of [12, Proposition 5.5] we have

$$WF_{\mathscr{F}L^q_{(\omega)}}(f) = WF_{M^{p,q}_{(\omega)}}(f), \tag{0.1}$$

for each $p,q\in[1,\infty]$, distribution f and appropriate weight function ω.

In the present paper we introduce discrete versions of $WF_{\mathscr{F}L^q_{(\omega)}}(f) = WF_{M^{p,q}_{(\omega)}}(f)$, denoted by $DF_{\mathscr{F}L^q_{(\omega)}}(f)$ and $DF_{M^{p,q}_{(\omega)}}(f)$ respectively, and prove that indeed (0.1) can be extended into

$$WF_{\mathscr{F}L^q_{(\omega)}}(f) = WF_{M^{p,q}_{(\omega)}}(f) = DF_{\mathscr{F}L^q_{(\omega)}}(f) = DF_{M^{p,q}_{(\omega)}}(f). \tag{0.1$'$}$$

1. Preliminaries

In this section we recall some notations and basic results. The proofs are in general omitted. We start by introducing some notations. In what follows we let Γ denote an open cone in $\mathbf{R}^d\setminus 0$. If $\xi\in\mathbf{R}^d\setminus 0$ is fixed, then an open cone which contains ξ is sometimes denoted by Γ_ξ.

Assume that ω and v are positive and measurable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$\omega(x+y)\le C\omega(x)v(y) \tag{1.1}$$

for some constant C which is independent of $x,y\in\mathbf{R}^d$. If v in (1.1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. If $\omega(x,\xi)\in\mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation.

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi)=\widehat{f}(\xi)\equiv(2\pi)^{-d/2}\int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,dx$$

when $f\in L^1(\mathbf{R}^d)$. We recall that $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Assume that $q\in[1,\infty]$ and $\omega\in\mathscr{P}(\mathbf{R}^{2d})$. Then the (weighted) Fourier Lebesgue space $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ is the Banach space which consists

of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathscr{F}L^q_{(\omega)}} = \|f\|_{\mathscr{F}L^q_{(\omega),x}} \equiv \|\widehat{f} \cdot \omega(x,\cdot)\|_{L^q}. \tag{1.2}$$

is finite. If $\omega = 1$, then the notation $\mathscr{F}L^q$ is used instead of $\mathscr{F}L^q_{(\omega)}$.

Remark 1.1. Here as in [12] we remark that it might seem to be strange that we permit weights $\omega(x,\xi)$ in (1.2) that are dependent on both x and ξ, though $\widehat{f}(\xi)$ only depends on ξ. The reason is that later on it will be convenient to permit such x dependency. We note however that the fact that ω is v-moderate for some $v \in \mathscr{P}(\mathbf{R}^{2d})$ implies that different choices of x give rise to equivalent norms. Therefore, the condition $\|f\|_{\mathscr{F}L^q_{(\omega),x}} < \infty$ is independent of x.

The modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \Big(\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} |V_\varphi f(x,\xi)\omega(x,\xi)|^p\, dx \Big)^{q/p} d\xi \Big)^{1/q} \tag{1.3}$$

is finite. Here $V_\varphi f$ is the short-time Fourier transform of f with respect to $\varphi \in \mathscr{S}(\mathbf{R}^d)\setminus 0$, which is equal to $\mathscr{F}(f\,\overline{\varphi(\,\cdot\,-x)})(\xi)$. We note that $V_\varphi f$ takes the form

$$V_\varphi f(x,\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\overline{\varphi(y-x)} e^{-i\langle y,\xi\rangle}\, dy$$

when $f \in L^1(\mathbf{R}^d)$.

If $\Gamma \subseteq \mathbf{R}^d \setminus 0$ is an open cone, then we let $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}}$ and $|f|_{M^{p,q,\Gamma}_{(\omega)}}$ be the seminorms

$$|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}} \equiv \Big(\int_\Gamma |\widehat{f}(\xi)\omega(x,\xi)|^q\, d\xi \Big)^{1/q} \tag{1.4}$$

and

$$|f|_{M^{p,q,\Gamma}_{(\omega)}} \equiv \Big(\int_\Gamma \Big(\int_{\mathbf{R}^d} |V_\varphi f(x,\xi)\omega(x,\xi)|^p\, dx \Big)^{q/p} d\xi \Big)^{1/q}. \tag{1.5}$$

respectively. Here we note that these semi-norms might attain the value $+\infty$.

Assume now that $X \subseteq \mathbf{R}^d$ is open. The wave-front set $WF_{\mathscr{F}L^q_{(\omega)}}(f)$ of $f \in \mathscr{D}'(X)$ consists of all pairs $(x_0,\xi_0) \in X \times (\mathbf{R}^d \setminus 0)$ such that for each $\chi \in C_0^\infty(X)$ with $\chi(x_0) \neq 0$ and each conical neighbourhood Γ of ξ_0 it holds

$$|\chi f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}} = +\infty.$$

In the same way, the wave-front set $WF_{M^{p,q}_{(\omega)}}(f)$ of $f \in \mathscr{D}'(X)$ consists of all pairs $(x_0,\xi_0) \in X \times (\mathbf{R}^d \setminus 0)$ such that for each $\chi \in C_0^\infty(X)$ with $\chi(x_0) \neq 0$ and each conical neighbourhood Γ of ξ_0 it holds

$$|\chi f|_{M^{p,q,\Gamma}_{(\omega)}} = +\infty.$$

2. Discrete semi-norms in Fourier-Lebesgue spaces

In this section we introduce discrete analogues of the non-discrete seminorms (1.4) and (1.5). We also show that these semi-norms are finite, if and only if corresponding non-discrete semi-norms are finite.

Assume that $q \in [1, \infty]$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and that $H \subseteq \mathbf{R}^d$ is a discrete set. Then we set

$$|f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(H)} = |f|^{(D)}_{\mathscr{F}L^q_{x,(\omega)}(H)} \equiv \Big(\sum_{\{\xi_k\} \in H} |\widehat{f}(\xi_k)\omega(x, \xi_k)|^q \Big)^{1/q}$$

(with obvious modifications when $q = \infty$). As in the continuous case, we note that the condition

$$|f|^{(D)}_{\mathscr{F}L^q_{x,(\omega)}(H)} < \infty$$

is independent of $x \in \mathbf{R}^d$. From now on we assume that ω is independent of x.

Lemma 2.1. *Assume that Γ and Γ_0 are open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_0} \subseteq \Gamma$, and that $\Lambda \subseteq \mathbf{R}^d$ is a lattice. Also assume that $f \in \mathscr{E}'(\mathbf{R}^d)$ and $\omega \in \mathscr{P}(\mathbf{R}^d)$. If $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}}$ is finite, then $|f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Gamma_0 \cap \Lambda)}$ is finite.*

Proof. We only prove the result for $q < \infty$, leaving the small modifications in the case $q = \infty$ for the reader. Assume that $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}} < \infty$, and let $H = \Gamma \cap \Lambda$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$ be such that $\varphi = 1$ in $\operatorname{supp} f$. Then

$$(|f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Gamma_0 \cap \Lambda)})^q = \sum_{\{\xi_k\} \in H} |\mathscr{F}(\varphi f)(\xi_k)\omega(\xi_k)|^q$$

$$= (2\pi)^{-qd/2} \sum_{\{\xi_k\} \in H} \Big| \int \widehat{\varphi}(\xi_k - \eta)\widehat{f}(\eta)\omega(\xi_k)\, d\eta \Big|^q \le (2\pi)^{-qd/2}(S_1 + S_2),$$

where

$$S_1 = \sum_{\{\xi_k\} \in H} \Big| \int_\Gamma \widehat{\varphi}(\xi_k - \eta)\widehat{f}(\eta)\omega(\xi_k)\, d\eta \Big|^q,$$

$$S_2 = \sum_{\{\xi_k\} \in H} \Big| \int_{\complement\Gamma} \widehat{\varphi}(\xi_k - \eta)\widehat{f}(\eta)\omega(\xi_k)\, d\eta \Big|^q.$$

We need to estimate S_1 and S_2. Let $v \in \mathscr{P}(\mathbf{R}^d)$ be chosen such that ω is v-moderate. By Minkowski's inequality we get

$$S_1 \le \sum_{\{\xi_k\}\in H} \Big(\int_\Gamma |\widehat{\varphi}(\xi_k - \eta)v(\xi_k - \eta)||\widehat{f}(\eta)\omega(\eta)|\, d\eta\Big)^q$$

$$= \sum_{\{\xi_k\}\in H} \Big(\int_\Gamma |\widehat{\varphi}(\xi_k-\eta)v(\xi_k-\eta)|^{1/q'}(|\widehat{\varphi}(\xi_k-\eta)v(\xi_k-\eta)|^{1/q}|\widehat{f}(\eta)\omega(\eta)|)\, d\eta\Big)^q$$

$$\le \|\widehat{\varphi}\omega\|_{L^1}^{q/q'} \sum_{\{\xi_k\}\in H} \int_\Gamma |\widehat{\varphi}(\xi_k - \eta)v(\xi_k - \eta)||\widehat{f}(\eta)\omega(\eta)|^q\, d\eta$$

$$\le C \int_\Gamma |\widehat{f}(\eta)\omega(\eta)|^q\, d\eta = C|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}},$$

where

$$C = \|\widehat{\varphi}\omega\|_{L^1}^{q/q'} \sup_{\eta\in\mathbf{R}^d} \sum_{\{\xi_k\}\in H} |\widehat{\varphi}(\xi_k - \eta)v(\xi_k - \eta)| < \infty.$$

This proves that S_1 is finite.

It remains to prove that S_2 is finite. We observe that $|\xi_k - \eta| \ge c\max(|\xi_k|, |\eta|)$ when $\xi_k \in H$ and $\eta \in \complement\Gamma$, and since f has compact support it follows that $|\widehat{f}(\xi)| \le C\langle\xi\rangle^{N_0}$ for some positive constants C and N_0. Furthermore, since $\varphi \in C_0^\infty$, it follows that for each $N \ge 0$, there is a constant C_N such that $|\widehat{\varphi}(\xi)| \le C_N\langle\xi\rangle^{-N}$. This gives

$$S_2 \le C_1 \sum_{\{\xi_k\}\in H} \Big(\int_{\complement\Gamma} \langle\xi_k - \eta\rangle^{-N}\langle\eta\rangle^{N_0}\, d\eta\Big)^q$$

$$\le C_2 \sum_{\{\xi_k\}\in H} \Big(\int \langle\xi_k\rangle^{-N/2}\langle\eta\rangle^{N_0-N/2}\, d\eta\Big)^q,$$

for some constants C_1 and C_2. The result now follows, since the right-hand side is finite when N is chosen larger than $2(N_0 + d)$. The proof is complete. $\square$

In the next result we prove a converse to Lemma 2.1, in the case that the lattice Λ is dense enough. Let $e_1, \dots, e_d$ in $\mathbf{R}^d$ be a basis for Λ, i. e. for some $x_0 \in \Lambda$ we have

$$\Lambda = \{\, x_0 + t_1e_1 + \cdots + t_de_d\, ;\, t_1, \dots, t_d \in \mathbf{Z}\,\}.$$

Then the parallelepiped, spanned by $e_1, \dots, e_d$ and with corners in Λ is called a Λ-*parallelepiped*. We let $\mathcal{A}(\Lambda)$ be the set of all Λ-parallelepipeds. For future references we note that if $D_1, D_2 \in \mathcal{A}(\Lambda)$, then their volumes $|D_1|$ and $|D_2|$ agree, and for conveniency we let $\|\Lambda\|$ denote the common value, i. e.

$$\|\Lambda\| = |D_1| = |D_2|.$$

Assume that Λ_1 and Λ_2 are lattices in $\mathbf{R}^d$ with bases $e_1, \dots, e_d$ and $\varepsilon_1, \dots, \varepsilon_d$ respectively. Then the pair (Λ_1, Λ_2) is called an *admissible lattice pair*, if for some $0 < c \le 2\pi$ we have $\langle e_j, \varepsilon_j \rangle = c$ and $\langle e_j, \varepsilon_k \rangle = 0$ when $j \neq k$. If in addition $c < 2\pi$, then (Λ_1, Λ_2) is called a *strongly admissible lattice pair*. If instead $c = 2\pi$, then the pair (Λ_1, Λ_2) is called a *weakly admissible lattice pair*

Lemma 2.2. *Assume that $q \in [1, \infty]$, (Λ_1, Λ_2) is a strongly admissible lattice pair, $K \in \mathcal{A}(\Lambda_1)$, and that $f \in \mathscr{E}'(\mathbf{R}^d)$ is such that an open neighbourhood of its support is contained in K. Also assume that Γ and Γ_0 are open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_0} \subseteq \Gamma$. If $|f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Gamma \cap \Lambda_2)}$ is finite, then $|f|_{\mathscr{F}L^{q,\Gamma_0}_{(\omega)}}$ is finite.*

Proof. We shall use similar arguments as in the proof of Lemma 2.1. Again we prove the result only for $q < \infty$. The small modifications to the case $q = \infty$ is left for the reader. Assume that $|f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Gamma \cap \Lambda_2)} < \infty$, and let $\varphi \in C_0^\infty(K)$ be equal to one in the support of f. By expanding $f = \varphi f$ into a Fourier series on K we get

$$\widehat{f}(\xi) = C \sum_{\xi_k \in \Lambda_2} \widehat{\varphi}(\xi - \xi_k) \widehat{f}(\xi_k),$$

where the positive constant C only depends on Λ_2. If $H_1 = \Lambda_2 \cap \Gamma$ and $H_2 = \Lambda_2 \cap \complement \Gamma$, then

$$\int_{\Gamma_0} |\widehat{f}(\xi)\omega(\xi)|^q \, d\xi = C^q \int_{\Gamma_0} \Big| \sum_{\xi_k \in \Lambda_2} \widehat{\varphi}(\xi - \xi_k) \widehat{f}(\xi_k) \omega(\xi) \Big|^q \, d\xi$$

$$\le C^q (S_1 + S_2),$$

where

$$S_1 = \int_{\Gamma_0} \Big| \sum_{\{\xi_k\} \in H_1} \widehat{\varphi}(\xi - \xi_k) \widehat{f}(\xi_k) \omega(\xi_k) \omega(\xi) \Big|^q \, d\xi,$$

$$S_2 = \int_{\Gamma_0} \Big| \sum_{\{\xi_k\} \in H_2} \widehat{\varphi}(\xi - \xi_k) \widehat{f}(\xi_k) \omega(\xi_k) \omega(\xi) \Big|^q \, d\xi.$$

We need to estimate S_1 and S_2. Let $v \in \mathscr{P}(\mathbf{R}^d)$ be chosen such that ω is v-moderate. By Minkowski's inequality we get

$$S_1 = \int_{\Gamma_0} \Big(\sum_{\{\xi_k\} \in H_1} |\widehat{\varphi}(\xi - \xi_k) v(\xi - \xi_k)||\widehat{f}(\xi_k)\omega(\xi_k)| \Big)^q \, d\xi$$

$$\le C_1 \int_{\Gamma_0} \Big(\sum_{\{\xi_k\} \in H_1} |\widehat{\varphi}(\xi - \xi_k) v(\xi - \xi_k)||\widehat{f}(\xi_k)\omega(\xi_k)|^q \Big) \, d\xi,$$

$$\le C_2 \sum_{\{\xi_k\} \in H_1} |\widehat{f}(\xi_k)\omega(\xi_k)|^q$$

where C_1 is a constant and

$$C_2 = C_1 \|\varphi\|_{\mathscr{F}L^1_{(v)}} < \infty.$$

This proves that S_1 is finite.

It remains to prove that S_2 is finite. We observe that

$$|\xi - \xi_k| \ge c \max(|\xi|, |\xi_k|) \quad \text{when} \quad \xi \in \Gamma_0 \quad \text{and} \quad \xi_k \in H_2.$$

Furthermore, $|\widehat{f}(\xi_k)| \le C\langle \xi_k \rangle^{N_0}$ for some constants C and N_0, and for each $N \ge 0$, there is a constant C_N such that $|\widehat{\varphi}(\xi)| \le C_N \langle \xi \rangle^{-N}$. This gives

$$S_2 \le C_1 \int_{\Gamma_0} \Big(\sum_{\{\xi_k\} \in H_2} \langle \xi - \xi_k \rangle^{-N} \langle \xi_k \rangle^{N_0} \Big)^q \, d\xi$$

$$\le C_2 \int_{\Gamma_0} \Big(\sum_{\{\xi_k\} \in H_2} \langle \xi \rangle^{-N/2} \langle \xi_k \rangle^{N_0 - N/2} \Big)^q \, d\xi$$

for some constants C_1 and C_2. The result now follows, since the right-hand side is finite when N is chosen larger than $2(N_0 + d)$. The proof is complete. □

Corollary 2.3. *Assume that $q \in [1, \infty]$, (Λ_1, Λ_2) is a strongly admissiblle lattice pair, $K \in \mathcal{A}(\Lambda_1)$, and that $f \in \mathscr{E}'(\mathbf{R}^d)$ is such that an open neighbourhood of its support is contained in K. Also assume that Γ and Γ_0 are open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_0} \subseteq \Gamma$. If $|f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Gamma \cap \Lambda_2)}$ is finite, then $|\chi f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Gamma_0 \cap \Lambda_2)}$ is finite.*

For the proof we recall that $|\chi f|_{\mathscr{F}L^{q,\Gamma_0}_{(\omega)}}$ is finite when $f \in \mathscr{E}'(\mathbf{R}^d)$, $\chi \in \mathscr{S}(\mathbf{R}^d)$, Γ_0, Γ are open cones such that $\overline{\Gamma_0} \subseteq \Gamma$ and $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}}$ is finite (cf. (2.3) in [12]).

Proof. Let Γ_1, Γ_2 be open cones such that $\overline{\Gamma_j} \subseteq \Gamma_{j+1}$ and $\overline{\Gamma_2} \subseteq \Gamma$, $j = 0, 1$, and assume that $|f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Gamma \cap \Lambda_2)} < \infty$. Then Lemma 2.2 shows

that $|f|_{\mathscr{F}L^{q,\Gamma_2}_{(\omega)}}$ is finite. Hence (2.3) in [12] shows that $|\chi f|_{\mathscr{F}L^{q,\Gamma_1}_{(\omega)}} < \infty$, which implies that $|\chi f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Gamma_0\cap\Lambda_2)} < \infty$, in view of Lemma 2.1. The proof is complete. □

3. Admissible Gabor pairs

In this section we introduce the notion of admissible Gabor pairs (AGP) and provide examples which illustrates that conditions in Definition 3.1 are quite general.

Assume that $e_1,\dots,e_d$ is a basis for Λ_1, and that (Λ_1,Λ_2) is a weakly admissible lattice pair. If $f\in L^2_{loc}$ is periodic with respect to Λ_1, and D is the parallelepiped, spanned by $\{e_1,\dots,e_d\}$, then we may make Fourier expantion of f as

$$f=\sum_{\{\xi_k\}\in\Lambda_2} c_k e^{i\langle\,\cdot\,,\xi_k\rangle} \tag{3.1}$$

(with convergence in L^2_{loc}), where the coefficients c_k are given by

$$c_k=|D|^{-1}\int_D f(y)e^{-i\langle y,\xi_k\rangle}\,dy. \tag{3.2}$$

We note that if instead $f\in L^2$ is supported in D, then (3.1) is still true in D, and the constant c_k can in this situation be written as

$$c_k=(2\pi)^{-d}\|\Lambda_2\|\int f(y)e^{-i\langle y,\xi_k\rangle}\,dy. \tag{3.2$'$}$$

For non-periodic functions and distributions we instead make Gabor expansions. More precisely, let (Λ_1,Λ_2) be a strongly admissible lattice pair, with $\Lambda_1=\{x_j\}_{j\in J}$ and $\Lambda_2=\{\xi_k\}_{k\in J}$. Also let

$$\begin{aligned} \varphi,\psi\in C_0^\infty(\mathbf{R}^d),\quad &\varphi_{j,k}(x)=\varphi(x-x_j)e^{i\langle x,\xi_k\rangle}\\ \text{and}\quad &\psi_{j,k}(x)=\psi(x-x_j)e^{i\langle x,\xi_k\rangle} \end{aligned} \tag{3.3}$$

be such that $\{\varphi_{j,k}\}_{j,k\in J}$ and $\{\psi_{j,k}\}_{j,k\in J}$ are dual Gabor frames (see [7] for the definition of Gabor frames and their duals). If $f\in\mathscr{S}'(\mathbf{R}^d)$, then

$$f=\sum_{j,k\in J} c_{j,k}\varphi_{j,k}, \tag{3.4}$$

where

$$c_{j,k}=(f,\psi_{j,k})_{L^2(\mathbf{R}^d)} \tag{3.5}$$

and the constant $C_{\varphi,\psi}$ depends on the frames only. Here the serie converges in $\mathscr{S}'(\mathbf{R}^d)$.

By replacing the lattices Λ_1 and Λ_2 here above with $\varepsilon\Lambda_1$ and Λ_2/ε, and φ and ψ with $\varphi^\varepsilon=\varphi(\,\cdot\,/\varepsilon)$ and $\psi^\varepsilon=\psi(\,\cdot\,/\varepsilon)$ respectively, we still have

$$f=\sum_{j,k\in J} c_{j,k}(\varepsilon)\varphi^\varepsilon_{j,k}, \tag{3.4$'$}$$

where

$$c_{j,k}(\varepsilon) = C_{\varphi,\psi}(\varepsilon)(f, \psi^\varepsilon_{j,k})_{L^2(\mathbf{R}^d)}, \tag{3.5$'$}$$

and

$$\varphi^\varepsilon_{j,k} = \varphi_{j,k}(\,\cdot\,/\varepsilon), \quad \psi^\varepsilon_{j,k} = \psi_{j,k}(\,\cdot\,/\varepsilon). \tag{3.6}$$

Here the constants $C_{\varphi,\psi}(\varepsilon)$ depends on φ, ψ and ε.

In some situations it is convenient to play with the parameter ε in $\varepsilon\Lambda_1$, $\varphi^\varepsilon = \varphi(\,\cdot\,/\varepsilon)$ and $\psi^\varepsilon = \psi(\,\cdot\,/\varepsilon)$, but keeping Λ_2 fixed and independent of ε. A problem is then that (3.4)$'$ and (3.5)$'$ might be violated. In the following we establish sufficient conditions for this to work properly. We first introduce *admissible Gabor pairs*.

Definition 3.1. Assume that $\varepsilon \in (0,1]$, $\{x_j\}_{j\in J} = \Lambda_1 \subseteq \mathbf{R}^d$ and $\{\xi_k\}_{k\in J} = \Lambda_2 \subseteq \mathbf{R}^d$ are lattices and let $\Lambda_1(\varepsilon) = \varepsilon\Lambda_1$. Also let $\varphi, \psi \in C_0^\infty(\mathbf{R}^d)$ be non-negative, and set

$$\begin{aligned} \varphi^\varepsilon &= \varphi(\,\cdot\,/\varepsilon), & \psi^\varepsilon &= \psi(\,\cdot\,/\varepsilon), \\ \varphi^\varepsilon_{j,k} &= \varphi^\varepsilon(\,\cdot\, - \varepsilon x_j)e^{i\langle\,\cdot\,,\xi_k\rangle}, & \psi^\varepsilon_{j,k} &= \psi^\varepsilon(\,\cdot\, - \varepsilon x_j)e^{i\langle\,\cdot\,,\xi_k\rangle}, \end{aligned} \tag{3.7}$$

when $\varepsilon x_j \in \Lambda_1(\varepsilon)$ (i. e. $x_j \in \Lambda_1$) and $\xi_k \in \Lambda_2$. Then the pair $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ is called an *admissible Gabor pair* (AGP) if for each $\varepsilon \in (0,1]$, the sets $\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}$ and $\{\psi^\varepsilon_{j,k}\}_{j,k\in J}$ are dual Gabor frames.

By Definition 3.1 and Chapters 5–13 in [7] it follows that if $f \in \mathscr{S}'(\mathbf{R}^d)$, (3.7) is fulfilled and $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ is an AGP, then

$$f = \sum_{j,k\in J} c_{j,k}(\varepsilon)\varphi^\varepsilon_{j,k}, \tag{3.4$''$}$$

for every $\varepsilon \in (0,1]$, where

$$c_{j,k}(\varepsilon) = (f, \psi^\varepsilon_{j,k})_{L^2(\mathbf{R}^d)} \tag{3.5$''$}$$

Furthermore, from the investigations in [7] it follows that (Λ_1, Λ_2) in Definition 3.1 should be a strongly admissible lattice pair, if $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ might be an AGP.

In the following lemma we prove that if Λ_1 and Λ_2 are the same as in Definition admgaborframe, $\{\varphi_{j,k}\}_{j,k\in J}$ and $\{\psi_{j,k}\}_{j,k\in J}$ are dual Gabor frames which satisfy

$$\sum_{x_j\in\Lambda_1} \varphi(\,\cdot\, - x_j)\psi(\,\cdot\, - x_j) = (2\pi)^{-d}\|\Lambda_2\|, \tag{3.8}$$

and $\varphi^\varepsilon_{j,k}$ and $\psi^\varepsilon_{j,k}$ are given by (3.7), then $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ is an admissible Gabor pair.

Remark 3.2. If $\varphi = \psi$, then (3.8) describes the tight frame property of the corresponding Gabor frame, cf. [7, Theorem 6.4.1].

Proposition 3.3. *Assume that* $0 < \varepsilon \le 1$*,* $\varphi, \psi \in C_0^\infty(\mathbf{R}^d)$ *are non-negative,* $\varphi_{j,k}$*,* $\psi_{j,k}$*,* $\varphi_{j,k}^\varepsilon$ *and* $\psi_{j,k}^\varepsilon$ *are given by* (3.3) *and* (3.7)*. Also assume that* $\{\varphi_{j,k}\}_{j,k\in J}$ *and* $\{\psi_{j,k}\}_{j,k\in J}$ *are dual Gabor frames, and that* (3.8) *holds. Then* $(\{\varphi_{j,k}^\varepsilon\}_{j,k\in J}, \{\psi_{j,k}^\varepsilon\}_{j,k\in J})$ *is an admissible Gabor pair.*

Proof. We shall prove that $\{\varphi_{j,k}^\varepsilon\}_{j,k\in J}$ and $\{\psi_{j,k}^\varepsilon\}_{j,k\in J}$ are dual Gabor frames for each $\varepsilon \in (0,1]$. This is obviously true when $\varepsilon = 1$.

Assume that $f \in C_0^\infty(\mathbf{R}^d)$ and that ε is small enough such that the supports of $\varphi_{j,k}^\varepsilon$ and $\psi_{j,k}^\varepsilon$ are contained in a parallelepiped D, spanned by the basis for the dual frame of Λ_2. We shall prove that

$$h_\varepsilon(x) = \sum_{j,k\in J} c_{j,k}(\varepsilon)\varphi_{j,k}^\varepsilon(x)$$

is equal to $f(x)$ when $c_{j,k}(\varepsilon)$ is given by $(3.5)''$. For conveniency we let Θ be the right-hand side of (3.8), i.e. $\Theta = (2\pi)^{-d}\|\Lambda_2\|$. By the inversion formula for Fourier series (cf. (3.1) and $(3.2)'$), we get

$$\begin{aligned} h_\varepsilon(x) = \Theta^{-1}\sum_{j\in J}\Big(\sum_{k\in J}\Theta\int f(y)\psi^\varepsilon(y-\varepsilon x_j)e^{-i\langle y,\xi_k\rangle}\,dy\,e^{i\langle x,\xi_k\rangle}\Big)\varphi^\varepsilon(x-\varepsilon x_j) \\ = \Theta^{-1}\sum_{j\in J} f(x)\varphi^\varepsilon(x-\varepsilon x_j)\psi^\varepsilon(x-\varepsilon x_j) = f(x), \end{aligned}$$

where the last equality follows from (3.8). This proves the result for small ε and $f \in C_0^\infty$.

Next assume that $\varepsilon \in (0,1]$ is arbitrary and consider again h_ε. Since $f, \varphi, \psi \in C_0^\infty$, it follows that $\widehat{f}, \widehat{\varphi}, \widehat{\psi}$ are entire functions which turn rapidly to zero at infinity on $\mathbf{R}^d$. This implies that

$$\begin{aligned} \kappa_{1,j,k}(\varepsilon,\zeta) &= \mathscr{F}(\psi_{j,k}^\varepsilon)(\zeta) = \varepsilon^d e^{i\varepsilon\langle x_j,\xi_k-\zeta\rangle}\widehat{\psi}(\varepsilon(\zeta-\xi_k)) \quad \text{and} \\ \kappa_{2,j,k}(\varepsilon,\zeta) &= \mathscr{F}(\varphi_{j,k}^\varepsilon)(\zeta) = \varepsilon^d e^{i\varepsilon\langle x_j,\xi_k-\zeta\rangle}\widehat{\varphi}(\varepsilon(\zeta-\xi_k)), \end{aligned}$$

are real-analytic in ε. This implies that

$$\begin{aligned} \kappa_3(\varepsilon,\zeta) \equiv \widehat{h}_\varepsilon(\zeta) = (2\pi)^{-d/2}\sum_{j,k\in J}\kappa_{2,j,k}(\varepsilon,\zeta)\mathscr{F}(f\psi_{j,k}^\varepsilon)(\xi_k) \\ = (2\pi)^{-d}\sum_{j,k\in J}\kappa_{2,j,k}(\varepsilon,\zeta)\big(\widehat{f} * \kappa_{1,j,k}(\varepsilon,\cdot)\big)(\xi_k) \end{aligned}$$

is real analytic in ε.

A combination of the latter real analyticity property and the fact that $\kappa_3(\varepsilon,\xi) = \hat{f}(\xi)$ when $\varepsilon = 1$ or ε is small enough, shows that $\mathscr{F}^{-1}(\kappa_3(\varepsilon,\cdot))(x) = f(x)$ for all $\varepsilon \in (0,1]$. This proves the result in the case $f \in C_0^\infty(\mathbf{R}^d)$. For general $f \in L^2(\mathbf{R}^d)$, the result now follows from the fact that C_0^∞ is dense in L^2. The proof is complete. $\square$

Example 3.4. Let $\alpha, \beta \in \mathbf{R}_+$ be such that $\alpha \cdot \beta < 2\pi$, $\Lambda_1 = \{x_j\}_{j\in J} = \alpha \mathbf{Z}^d$ and $\Lambda_2 = \{\xi_k\}_{k\in J} = \beta \mathbf{Z}^d$. Also let Q_1 and Q_2 be cubes with centers at origin and side-length's α_1 and α_2 respectively, and such that

$$\alpha < \alpha_1 < \alpha_2 = 2\pi/\beta,$$

and choose $\varphi \in C_0^\infty(Q_2)$ and $\psi \in C_0^\infty(Q_1)$ such that $\varphi = 1$ on $\operatorname{supp}\psi$ and

$$\sum_{j\in J} \psi(\cdot - x_j) = \Big(\frac{\beta}{2\pi}\Big)^d.$$

By expanding $f\cdot\psi(\cdot - x_j)$ in Fourier series in $x_j + Q_2$ for each $j \in J$, it follows that $\{\varphi_{j,k}\}_{j,k\in J}$ and $\{\psi_{j,k}\}_{j,k\in J}$ in (3.3) are dual Gabor frames. Therefore, (3.4) and (3.5) holds.

By Proposition 3.3 it now follows that $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ is an admissible Gabor pair.

Remark 3.5. Assume that $p, q \in [1,\infty]$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $f \in \mathscr{S}'(\mathbf{R}^d)$, $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ is an admissible Gabor pair, and that $(3.4)'$ and $(3.5)'$ hold. Then it follows that $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$, if and only if

$$\|f\|_{[\varepsilon]} \equiv \Big(\sum_{k\in J}\Big(\sum_{j\in J} |c_{j,k}(\varepsilon)\omega(\varepsilon x_j, \xi_j)|^p\Big)^{q/p}\Big)^{1/q}$$

if finite. Furthermore, for every $\varepsilon \in (0,1]$, the norm $f \mapsto \|f\|_{[\varepsilon]}$ is equivalent to the modulation space norm (1.3). (Cf. [2–4, 7].)

4. Discrete wave-front sets - Fourier Lebesgue and modulation spaces

In this section we define discrete wave-front sets of Fourier Lebesgue and modulation space types, and prove that they agree with the corresponding wave-front sets of continuous type.

We start with the following definitions.

Definition 4.1. Assume that $\omega \in \mathscr{P}(\mathbf{R}^d)$, $f \in \mathscr{D}'(X)$, $x_0 \in X$, (Λ_1, Λ_2) is a strongly admissible lattice pair in $\mathbf{R}^d$ and that $\{\xi_k\}_{j\in J} = \Lambda_2$. Also assume that $D \in (\Lambda_1)$ contains x_0. Then the discrete wave-front set $DF_{\mathscr{F}L^q_{(\omega)}}(f)$ consists of all $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that for each $\chi \in C_0^\infty(D \cap X)$ with $\chi(x_0) \neq 0$ and each open conical neighbourhood Γ of ξ_0, it holds

$$|\chi f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(\Lambda)} = \infty.$$

For the definition of discrete wave-front sets of modulation space type, we consider admissible Gabor pairs $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$, $\varepsilon \in (0,1]$, and let

$$J_{x_0}(\varepsilon) = J_{x_0}(\varepsilon, \varphi, \psi) = J_{x_0}(\varepsilon, \varphi, \psi, \Lambda_1)$$

be the set of all $j \in J$ such that

$$x_0 \in \operatorname{supp} \varphi^\varepsilon_{j,k} \quad \text{or} \quad x_0 \in \operatorname{supp} \psi^\varepsilon_{j,k}$$

Definition 4.2. Assume that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $f \in \mathscr{D}'(X)$, $x_0 \in X$ and $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$, $\varepsilon \in (0,1]$, are admissible Gabor pairs with respect to the lattices Λ_1 and Λ_2 in $\mathbf{R}^d$. Then the discrete wave-front set $DF_{M^{p,q}_{(\omega)}}(f)$ consists of all $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that for each $\varepsilon \in (0,1]$ and each open conical neighbourhood Γ of ξ_0, it holds

$$\Big(\sum_{\{\xi_k\}\in\Gamma\cap\Lambda_2} \Big(\sum_{j\in J_{x_0}(\varepsilon)} |c_{j,k}(\varepsilon)\omega(\xi_k)|^p \Big)^{q/p}\Big)^{1/q} = \infty,$$

where $f = \displaystyle\sum_{j,k\in J} c_{j,k(\varepsilon)}\varphi^\varepsilon_{j,k}$, and $c_{j,k}(\varepsilon) = C_{\varphi,\psi}(f,\psi^\varepsilon_{j,k})_{L^2(\mathbf{R}^d)}$ and the constant $C_{\varphi,\psi}$ depends on the frames only.

Roughly speaking, $(x_0,\xi_0) \in DF_{M^{p,q}_{(\omega)}}(f)$ means that f is nt locally in $M^{p,q}_{(\omega)}$ in the direction ξ_0. This interpretation coincide with the following theorem which is our main result:

Theorem 4.3. *Assume that $X \subseteq \mathbf{R}^d$ is open, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $f \in \mathscr{D}'(X)$ and $p,q \in [1,\infty]$. Then $(0.1)'$ holds.*

Proof. By Proposition 5.5 in [12] and Lemmas 2.1 and 2.2, it follows that the first two equalities in $(0.1)'$ hold. The result therefore follows if we prove that $DF_{\mathscr{F}L^q_{(\omega)}}(f) = DF_{M^{p,q}_{(\omega)}}(f)$.

First assume that $(x_0,\xi_0) \notin DF_{\mathscr{F}L^q_{(\omega)}}(f)$, and choose $\chi \in C_0^\infty(X)$, an open neighbourhood $X_0 \subseteq X$ of x_0 and conical neighbourhoods Γ,Γ_0 of ξ_0 such that

$$\overline{\Gamma_0} \subseteq \Gamma, \quad \chi(x) = 1 \quad \text{when} \quad x \in X_0,$$

$$\text{and} \quad |\chi\, f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(H)} < \infty, \quad H = \Lambda_2 \cap \Gamma.$$

Now let $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ be an admissible Gabor pair and choose $\varepsilon \in (0,1]$ such that $\operatorname{supp}\varphi^\varepsilon_{j,k}$ and $\operatorname{supp}\psi^\varepsilon_{j,k}$ is contained in X_0 when $x_0 \in \operatorname{supp}\varphi^\varepsilon_{j,k}$ and $x_0 \in \operatorname{supp}\psi^\varepsilon_{j,k}$. Since

$$c_{j,k}(\varepsilon) = C(f,\psi^\varepsilon_{j,k})_{L^2(\mathbf{R}^d)} = \mathscr{F}(f\,\psi(\,\cdot\,/\varepsilon - x_j))(\xi_k),$$

it follows from these support properties that if $H_0 = \Lambda_2 \cap \Gamma_0$, then

$$\begin{aligned} &\Big(\sum_{\{\xi_k\}\in H_0} |\mathscr{F}(f\,\psi(\,\cdot\,/\varepsilon - x_j))(\xi_k)\omega(\xi_k)|^q \Big)^{1/q} \\ &\qquad = |f\,\psi(\,\cdot\,/\varepsilon - x_j)|^{(D)}_{\mathscr{F}L^q_{(\omega)}(H_0)} = |f\,\chi\psi(\,\cdot\,/\varepsilon - x_j)|^{(D)}_{\mathscr{F}L^q_{(\omega)}(H_0)}, \end{aligned} \tag{4.1}$$

when $j \in J_{x_0}(\varepsilon)$. Hence, by combining Corollary 2.3 with the facts that $J_{x_0}(\varepsilon)$ is finite and $|\chi f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(H)} < \infty$, it follows that the expressions in (4.1) are finite and

$$\Big(\sum_{\{\xi_k\}\in H_0} \Big(\sum_{j\in J_{x_0}(\varepsilon)} |\mathscr{F}(f\,\psi(\,\cdot\,/\varepsilon - x_j))(\xi_k)\omega(\xi_k)|^p \Big)^{q/p} \Big)^{1/q} < \infty.$$

This implies that $(x_0,\xi_0) \notin DF_{M^{p,q}_{(\omega)}}(f)$, and we have proved that $DF_{M^{p,q}_{(\omega)}}(f) \subseteq DF_{\mathscr{F}L^q_{(\omega)}}(f)$.

In order to prove the opposite inclusion we assume that $(x_0,\xi_0) \notin DF_{M^{p,q}_{(\omega)}}(f)$, and we choose $\varepsilon \in (0,1]$, admissible Gabor pair $(\{\varphi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J})$ and conical neighbourhoods Γ, Γ_0 of ξ_0 such that $\overline{\Gamma_0} \subseteq \Gamma$ and

$$\Big(\sum_{\{\xi_k\}\in H} \Big(\sum_{j\in J_{x_0}(\varepsilon)} |\mathscr{F}(f\,\psi(\,\cdot\,/\varepsilon - x_j))(\xi_k)\omega(\xi_k)|^p \Big)^{q/p} \Big)^{1/q} < \infty, \qquad (4.2)$$

when $H = \Lambda_2 \cap \Gamma$. Also choose $\chi, \phi \in C_0^\infty(X)$ such that $\chi(x_0) \neq 0$,

$$\phi(x) \sum_{j\in J_{x_0}(\varepsilon)} \varphi^\varepsilon_{j,k}(x) = 1, \quad \text{when} \quad x \in \operatorname{supp}\chi.$$

Since $J_{x_0}(\varepsilon)$ is finite, Hölder's inequality gives

$$\begin{aligned}
|\chi f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(H_0)} &= \Big| \sum_{j\in J_{x_0}(\varepsilon)} (\chi\phi)\,(f\,\psi(\,\cdot\,/\varepsilon - x_j)) \Big|^{(D)}_{\mathscr{F}L^q_{(\omega)}(H_0)} \\
&\le \Big(\sum_{\{\xi_k\}\in H_0} \Big(\sum_{j\in J_{x_0}(\varepsilon)} |\mathscr{F}((\chi\phi) f\,\psi(\,\cdot\,/\varepsilon - x_j))(\xi_k)\omega(\xi_k)| \Big)^{q} \Big)^{1/q} \\
&\le C\Big(\sum_{\{\xi_k\}\in H_0} \Big(\sum_{j\in J_{x_0}(\varepsilon)} |\mathscr{F}((\chi\phi) f\,\psi(\,\cdot\,/\varepsilon - x_j))(\xi_k)\omega(\xi_k)|^p \Big)^{q/p} \Big)^{1/q},
\end{aligned}$$

By Corollary 2.3 and (4.2) it now follows that the right-hand side in the last estimates is finite. Hence $|\chi f|^{(D)}_{\mathscr{F}L^q_{(\omega)}(H_0)} < \infty$, which shows that $(x_0,\xi_0) \notin DF_{\mathscr{F}L^q_{(\omega)}}(f)$, and we have proved that $DF_{\mathscr{F}L^q_{(\omega)}}(f) \subseteq DF_{M^{p,q}_{(\omega)}}(f)$. The proof is complete. $\square$

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